# Announcements

1) Note for Rudin are up online under "Resources" on CTools

# Taylor Series

Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

Suppose the radius of convergence is nonzero.

Then for all $x$ in the open interval determined by the radius of convergence,

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\vdots$$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-(k-1)) a_n x^{n-k}$$

$$f^{(k)}(0) = k!\, a_k$$

So $a_k = \dfrac{f^{(k)}(0)}{k!}$

$\forall \; k \geq 0.$

Definition: (Taylor series)

Suppose $f$ is infinitely differentiable on $(c-R, c+R)$ for a real number $c$, $R>0$ $(R=\infty \Rightarrow (-\infty,\infty))$. The Taylor series of $f$ centered at $c$ is

$$\boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n}$$

Reduction: We will take $c=0$ in what follows (MacLaurin series).

**Problem:** When is a given function $f$ equal to its Taylor series? Is the equality always true on some nondegenerate interval?

**Example 1:** (exponentials)

We know from Calc II that on $(-\infty, \infty)$, we are supposed to have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Why?

Both $e^x = f(x)$

and $g(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Satisfy the differential

equation

$$h(x) = h'(x)$$

$\forall\, x \in \mathbb{R}$.

Note that $g$ converges for all $x \in \mathbb{R}$ by the ratio test.

As a corollary, we obtain

$$\boxed{\lim_{n\to\infty} \frac{x^n}{n!} = 0}$$

$\forall\ x \in \mathbb{R}$.

Now if

$$h(x) = h'(x) \text{ and}$$

$$h(x) \neq 0 \quad \forall\, x \in \mathbb{R},$$

$$\frac{h'(x)}{h(x)} = 1.$$

Integrate (i.e. find an antiderivative) for either side.

We get $\ln(h(x)) = x + c$.

Exponentiating,

$$h(x) = e^{x+c} = e^c e^x$$

$\forall \; x \in \mathbb{R}$. Therefore,

$$g(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^c e^x$$

for some $c \in \mathbb{R}$.

Plugging in $x=0$,

$$1 = g(0) = e^{c}$$

$\Rightarrow$ $c=0$, and

hence

$$g(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^{x} = f(x)$$

$\forall\ x \in \mathbb{R}$.

Unfortunately this procedure doesn't always work out so well. We need some theory!

**Theorem**: (generalized Mean Value)

Suppose $f, g$ are differentiable on $(a,b)$ and continuous on $[a,b]$. Then $\exists\, c$, $a < c < b$,

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$(g'(c),\ g(b)-g(a) \neq 0)$

Proof:

Define

$$h(x) = \left(f(b) - f(a)\right)g(x) - \left(g(b) - g(a)\right)f(x)$$

$\forall\, x \in [a,b]$. Apply the Mean Value Theorem to $h$. Then $\exists\, c$, $a < c < b$, Such that

$$h'(c) = \frac{h(b)-h(a)}{b-a}$$

$$= \frac{\Big(\big(f(b)-f(a)\big)g(b) - \big(g(b)-g(a)\big)f(b)\Big) - \Big(\big(f(b)-f(a)\big)g(a) - \big(g(b)-g(a)\big)f(a)\Big)}{b-a}$$

$$= \frac{\big(f(b)-f(a)\big)\big(g(b)-g(a)\big) - \big(f(b)-f(a)\big)\big(g(b)-g(a)\big)}{b-a}$$

$$= 0$$

But

$$h'(c) = (f(b)-f(a))g'(c) - (g(b)-g(a))f'(c)$$

$$= 0, \text{ so}$$

$$f'(c)(g(b)-g(a)) = g'(c)(f(b)-f(a))$$

Provided the requisite terms are nonzero, we obtain the result. $\square$

**Theorem:** (Lagrange Remainder)

Let $f$ be infinitely differentiable on $(-R, R)$, $R>0$. Set

$$a_n = \frac{f^{(n)}(0)}{n!}$$ and let

$$E_N(x) = \left( f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n \right)$$

for $x \in (-R, R)$.

Then $\exists\, c$,

$|c| < |x|$, with

$$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1}$$

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proof:

Step 1: $E_N^{(k)}(0) = 0$

$\forall\, k \in \mathbb{N} \cup \{0\}$

$$E_N^{(k)}(x) = f^{(k)}(x) - \sum_{n=k}^{N} n(n-1)\cdots(n-(k-1))\, a_n x^{n-k}$$

$$E_N^{(k)}(0) = f^{(k)}(0) - k!\, a_k$$

$$= f^{(k)}(0) - \frac{\cancel{k!}\, f^{(k)}(0)}{\cancel{k!}} = 0$$

✓

Step 2: Given $x_0 > 0$, $\exists \, (x_n)_{n=1}^{N+1}$

with $x_n > x_{n+1} > 0 \;\; \forall \;\; 0 \leq n < N+1$

and $$\frac{E_N^{(n)}(x_n)}{x_n^{N+1-n}} = \frac{E_N^{(n+1)}(x_{n+1})}{(N+1-n)\, x_{n+1}^{N+1-(n+1)}}$$

$\forall \;\; 0 \leq n < N+1.$

Apply generalized MVT and use induction.

$\underline{n=0}$ Let $x = x_0 > 0$.

Apply the generalized MVT to the functions $E_N(x)$ and $x^{N+1}$ to obtain $x_1 \in (0, x_0)$ with

$$\frac{E_N'(x_1)}{N x_1^{N-1}} = \frac{E_N(x_0) - E_N(0)}{x_0^N}$$

$$= \frac{E_N(x_0)}{x_0^N} \text{ by Step 1}$$

Now assume $n > 0$

and suppose we know the

result for $n$.

Then $\exists\, x_{n+1} \in (0, x_n)$

with

$$\frac{E_N^{(n)}(x_n)}{x_n^{N+1-n}} = \frac{E_N^{(n+1)}(x_{n+1}) - E_N^{(n+1)}(0)}{(N+1-n)\, x_{n+1}^{N+1-(n+1)}}$$

$$= \frac{E_N^{(n+1)}(x_{n+1})}{(N+1-n)\, x_{n+1}^{N+1-(n+1)}}$$

again by Step 1.

Now iterating these equalities,

$$\frac{E_N(x_0)}{x_0^{N+1}} = \frac{E_N'(x_1)}{(N+1)(x_1^N)}$$

$$= \frac{E_N''(x_2)}{(N+1)N\, x_2^{N-1}}$$

$$\vdots$$

$$= \frac{E_N^{(N+1)}(x_{N+1})}{(N+1)!\, x_{N+1}^{(N+1)-(N+1)}}$$

$$= \frac{E_N^{(N+1)}(x_{N+1})}{(N+1)!}$$

But

$$E_N(x) = f(x) - \sum_{n=0}^{N} a_n x^n$$

$$\Rightarrow E_N^{(N+1)}(x) = f^{(N+1)}(x).$$

Setting $c = x_{N+1}$ and $x = x_0$,

$$\frac{E_N(x)}{x^{N+1}} = \frac{f^{(N+1)}(c)}{(N+1)!}$$

$$\Rightarrow E_N(x) = \frac{f^{(N+1)}(c)\, x^{N+1}}{(N+1)!} \quad \square$$

Example 2: $(\sin(x) = f(x))$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

The coefficients

come from the derivatives,

i.e., $\dfrac{f^{(n)}(0)}{n!} = a_n$

By Lagrange's Remainder Theorem,

$$\left| E_N(x) \right| = \left| \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1} \right|$$

$$\leq \frac{|x|^{N+1}}{(N+1)!}$$

since $\left| \frac{d^n}{dx^n} \sin(x) \right| \leq 1$.

$$\frac{|x|^{N+1}}{(N+1)!} \to 0 \quad \forall x \in \mathbb{R},$$

So the convergence

is established.

# A Counterexample

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$